Chapter 10: Fuzzy Sets for *GIS*

Section 10.0. *GIS* revisited

Throughout this book, the concept of an interval implicitly relies upon three objects: (1) a space *S* of musical elements between which intervals or 'directed distances' can be defined, (2) a group structure *IVLS* within which intervals can be considered apart from the specific elements that form them and combined or inverted via the group operation, and (3) a function *int* that assigns to any pair of musical elements in *S* an interval in *IVLS*. These three objects together form a *GIS*(*S*, *IVLS*, *int*). We explored musical contexts in which a *GIS* can be applied, i.e., the space of pitch-classes, harmonic ratios, rhythmic classes, durations, etc. The two most commonly used *GIS* are *p-space* and *pc-space*.

This book has explored specific applications of *GIS* theory to pitch-class theory (where the primary *GIS* is *pc-space*) and, in so doing, has 'filled in the gaps' between Lewin's work and the more traditional work of pitch-class theorists. By defining intervals between finite sets of musical elements, we generalized to an arbitrary *GIS* well known facts about *pc-space*. In fact, we embedded pitch-class theory into the more general theory of *GIS*. In so doing, we can view pitch-class theory with a degree of flexibility that allows a composer or analyst to define intervals in many different ways. It is precisely this flexibility that allows us to now address some criticisms leveled at pitch-class theory.

Section 10.1. The problem of equivalence in *pc-space*

What are the right criteria for defining equivalence among pitch-class sets? Usually equivalence is defined by the group of transpositions and inversions (i.e., *CANON* contains only transpositions and inversions). However, for some musical contexts it is not clear that equivalence should be determined in this way.[1]

Critics are correct in stating that a pitch-class set may not be arbitrarily substituted for its inversion, given the appropriate musical environment.[2] Indeed, the quality in

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1 Castine 1994: 64; Benjamin 1974: 181; Lerdahl 1989: 66
2 Castine 1994: 64, 72

sonority of a major triad is fundamentally different from that of a minor triad, although, as members of the same orbit, they are considered equivalent. In some contexts, it may be more appropriate to establish equivalence of pitch-class sets through means very different from the traditional transpositions and inversions.[3] The generalizing power of *GIS* theory places the responsibility for determining criteria of equivalence upon the shoulders of the analyst or composer. Given the specific musical context, the analyst or composer must determine those operations that will best define set-class equivalence. As a bare minimum, of course, the operations chosen must define a group (*CANON*). This allows the analyst or composer to 'combine' or 'undo' the operations on the musical parameters.

The problem of equivalence pertains not only to relating pitch-class sets under *CANON*, but also to the definitions of pitch-class and interval-class that compose the very foundation of pitch-class theory.[4] For example, critics argue that the specific pitches C_0 and C_1 are quite distinct in context, although they both belong to pitch-class C; so are the minor second and minor ninth intervals, although they both belong to interval-class 1.[5] These concerns are valid so long as one maintains the primacy of octave equivalence among pitches or intervals (and inversional equivalence among intervals in interval-classes). Indeed, critics state that there may be a unique representative pitch in a pitch-class, or a unique representative interval in an interval-class, that satisfies the appropriate musical context.[6]

GIS theory shows that the structural characteristics of *pc-space* can be carried over to many types of spaces. If octave equivalence (for pitches or intervals) seems inappropriate in some context, for example, then try using *p-space* to model the environment. Of course you lose certain properties that are good to have in *pc-space* (such as the arithmetic mod 12) but you gain the subtle distinctions between a pitch and its octave variants which, given the musical context, may be wholly needed.[7] Similarly, you can choose not to use interval classes on the interval group *IVLS*, thereby

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3 Benjamin 1974: 181-82
4 Ibid.: 185; Castine 1994: 71; Morris 1987: 35-36
5 "…we must not forget that a melodic leap of a minor ninth is a completely different gesture from that of a descending minor second…and this difference may well be musically significant." Castine 1994: 71.
6 Ibid.: 72
7 Morris 1987: 36

distinguishing an interval from its inversion. A rich musical analysis indeed, is one that would use various *GIS* for different aspects of the study.

Section 10.2. Fuzzy pitch-class theory

Unfortunately, *GIS* theory does not address the most salient problems of pitch-class theory:

(1) Each element of a pitch-class set has equal weight, and there is no provision to describe structural relationships among the elements,[8]

(2) There is no procedure for determining which pitch-classes in a piece of music get chosen for analysis (i.e. placed into pitch-class sets).[9]

These shall be called, respectively, the *hierarchy* and *segmentation* problems.

To attack the hierarchy problem in pitch-class theory, theorist Peter Silberman describes pitch-class sets as weighted sets and uses the language of fuzzy set theory to describe them.[10] Each pitch-class s is accompanied by its characteristic membership value, that specifies the relative strength of s in a given pitch-class set.[11] Membership values are numbers in the closed interval [0, 1]. A pitch-class set X in the ordinary sense is called *crisp*. Its membership function X only assumes values 0 and 1, and $X(s) = 1$ iff $s \in X$ in the ordinary sense. For a noncrisp fuzzy set X there will be elements $s \in S$ for which $0 < X(s) < 1$. The value $X(s)$ specifies the degree to which s belongs to X, or the importance we assign its membership. Since the membership function tells all about X, we identify fuzzy sets with their membership functions.[12]

Definition *Let* (S, $IVLS$, int) *be a GIS. Define a fuzzy subset X of S to be a function X*:

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[8] Lerdahl 1989: 66; Castine 1994: 73. "By implication, current atonal theory regards all pitches contained within a set as somehow equal. To the ear, however, this is not so. Even without a general scale of pitch stability, structural importance tends to be attributed to events perceived as salient at the musical surface." Lerdahl and Jackendoff 1983: 300.

[9] Lerdahl 1989: 66; Castine 1994: 73

[10] Silberman 1997: 6

[11] Ibid.

[12] "…no ambiguity results from this double use of the same symbol. Each fuzzy set is completely and uniquely determined by one particular membership function; consequently, symbols of membership functions may also be used as labels of the associated fuzzy sets." Klir 1995: 11.

$S \rightarrow [0, 1]$. *For each* $s \in S$, $X(s)$ *determines the membership value of s in X*. *When the GIS is pc-space*, *then X is called a fuzzy pitch-class set*.

Given two fuzzy subsets X and Y of S, the standard $\vee$ and $\wedge$ notation for max and min yields their fuzzy union and intersection respectively: $(X \vee Y)(s) = X(s) \vee Y(s)$ and $(X \wedge Y)(s) = X(s) \wedge Y(s)$, for $s \in S$. The fuzzy complement of X with respect to S is $\overline{X}(s) = 1 - X(s)$, for $s \in S$.

Example *Let the GIS*(S, *IVLS*, *int*) be *pc-space*. Define fuzzy sets X and Y such that

(1) $X(0) = .8$, $X(4) = .4$, and $X(\mathrm{s}) = 0$ for all $s \neq 0, 4$.

(2) $Y(0) = .5$, $Y(7) = .2$, and $Y(\mathrm{s}) = 0$ for all $s \neq 0, 7$

Then:

(a) The fuzzy union of X and Y is $(X \vee Y)(0) = X(0) \vee Y(0) = .8$, $(X \vee Y)(4) = X(4) \vee Y(4) = .4$, $(X \vee Y)(7) = X(7) \vee Y(7) = .2$, and $(X \vee Y)(s) = 0$ for all $s \neq 0, 4, 7$.

(b) The fuzzy intersection of X and Y is $(X \wedge Y)(0) = X(0) \wedge Y(0) = .5$ and $(X \wedge Y)(s) = 0$ for all $s \neq 0$.

(c) The fuzzy complement of X is $\overline{X}(0) = 1 - X(0) = .2$, $\overline{X}(4) = 1 - X(4) = .6$, $\overline{X}(s) = 1$ for all $s \neq 0, 4$.

Section 10.3. The core and alpha-cuts

Defining pitch-class sets as fuzzy sets addresses the hierarchy problem. Indeed, the membership values of elements in a pitch-class set *X* specifies their relative importance in *X*.[13]

If we view a whole section of music as a fuzzy set of pitch-classes and assign membership values to each pitch-class, then we can define level sets of pitch-classes that determine the structural support of that section. Those pitch-classes whose membership values equal 1 constitute the *core* of that musical section. Indeed, in fuzzy set theory, the

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13 Silberman 1997: 8

core of a fuzzy set X is the crisp set $\{s \in X \mid X(s) = 1\}$.[14] We can make various *alpha-cuts* in a section of music to determine those pitch-classes that help to support the core.[15] Alpha cuts are crisp sets of the form $Y_\alpha = \{s \in X \mid X(s) \geq \alpha\}$ where $\alpha \in [0, 1]$. Each level of an analysis for a piece of music can determine higher values for α, until a core set of pitch-classes for that particular piece is reached.[16]

In some music, such as sections of a few twelve tone works, an analysis may not provide a core set of pitch-classes. However most musical analyses will suggest some hierarchical relation among the pitch-classes. Indeed, much of the analysis of the music of Stravinsky, Bartok, Berg, and even Schoenberg utilizes a concept called *pitch-class centricity* that refers to a stable, referential collection of pitch-classes (the core set) that predominate throughout a section of music.[17]

Section 10.4. Orbits of fuzzy pitch-class sets

We can also consider orbits of fuzzy pitch-class sets. Note that the way we assign degrees of membership is open to consideration, and might or might not prove appropriate.

Definition *Let* $(S, IVLS, int)$ *be a GIS. Let CANON be a group of permutations of S. Let X be a fuzzy subset of S. Then the orbit of X is the set* $\Phi = \{f(X) \mid f \in CANON\}$ *where, for* $y \in S$, $f(X)(y) = X(f^{-1}(y))$. *If the GIS is pc-space, then* Φ *is called an orbit of a fuzzy pitch-class set.*

Example Consider the *GIS*$(S, IVLS, int)$ of *pc-space* with *CANON* equal to the group of transpositions. Let X be a fuzzy subset of S such that $X(0) = 1 > X(1) = .7 > X(3) = .5 > X(7) = .2 > X(s) = 0$ for $s \neq 0, 1, 3, 7$. Then, for example, $T_5(X)(5) = X(T_7(5)) = X(0) = 1 > T_5(X)(6) = .7 > T_5(X)(8) = .5 > T_5(X)(0) = .2 > T_5(X)(s) = 0$ for $s \neq 5, 6, 8, 0$.

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14 Klir 1995: 21
15 Ibid.: 19
16 Personal communication with Curtis Krueker.
17 "All tonal music is centric, focused on specific pitch classes or triads, but not all centric music is tonal. Even without the resources of tonality, music can be organized around referential centers. A great deal of

If we view a section of music as a fuzzy set X of orbits of pitch-class sets, we can assign orbits of pitch-class sets different membership values in X. Orbits of pitch-class sets can be ordered in many different ways, the K- or Kh-relationship is an example. We can use the language of fuzzy set theory to describe the K- or Kh-relationship; those orbits that are K-related to every other orbit in X are assigned a higher membership value than those orbits in X that are not. Yet fuzzy set theory allows for many different types of relationships other than just the K- or Kh-relationship.

Section 10.5. Fuzzy sets of intervals

We can also use fuzzy set theory to describe hierarchical relationships among intervals. Let S be a musical space, X be a fuzzy subset of S, and $X(s)$, $X(t) > 0$. We define the membership value of the interval from s to t to be the minimum of the membership values assigned to s and t in X. It is precisely the presence of s and t that allows us to hear the interval $int(s, t)$. For example, if pitch-classes C and D are deemed musically important, then so should the intervals formed from C to D or from D to C.[18] Note that the way we assign degrees of membership is open to consideration, and might or might not prove appropriate.

Definition *Let* $(S, IVLS, int)$ *be a GIS. Let* $\tilde{S}$ *be a fuzzy subset of S. Let* $\widetilde{IVLS}_{\tilde{S}}$ *be a fuzzy subset of IVLS such that, for* $j \in IVLS$,

$$\widetilde{IVLS}_{\tilde{S}}(j) = \begin{cases} k = \wedge\left\{\tilde{S}(s) \wedge \tilde{S}(t) \,\middle|\, s,t \in S \text{ and } int(s,t) = j\right\}, \text{ provided } k > 0 \\ 0, \text{ otherwise} \end{cases}$$

We define $\widetilde{IVLS}_{\tilde{S}}$ to be the fuzzy set of intervals induced by $\tilde{S}$.

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post-tonal music focuses on specific pitches, pitch-classes, or pitch-class sets as a way of shaping and organizing the music." Strauss 2000: 113-14.

[18] We could treat our group of intervals *IVLS* as a fuzzy group where, according to fuzzy group theorists, *IVLS* as a fuzzy set must satisfy the following conditions: (1) $IVLS(ij) \geq IVLS(i) \wedge IVLS(j)$ and (2) $IVLS(i^{-1}) \geq IVLS(i)$, for all $i, j \in IVLS$. Kumar 1993: 7. However, we run into musical problems by doing so. For example, in *pc-space*, for $2 \in IVLS = \mathbb{Z}_{12}$, we must have $IVLS(2) = IVLS(1 + 1) \geq IVLS(1) \wedge IVLS(1) = IVLS(1)$. This would imply that the membership value of the interval of a major second, 2, is necessarily at least the membership value of a minor second, 1. This conclusion is problematic if the membership value of a minor second is musically determined to be greater than that of a major second.

Example Let (*S*, *IVLS*, *int*) be the *GIS* of *pc-space*. Let *X* be a fuzzy pitch-class set where *X*(4) = 1 > *X*(0) = .5 > *X*(7) = .2 > *X*(s) = 0 for s ≠ 0, 4, 7. Then $\widetilde{IVLS}_X(j) = .5$ for *j* = 4, 8; $\widetilde{IVLS}_X(j) = .2$ for *j* = 0, 3, 5, 9, 7; $\widetilde{IVLS}_X(j) = 0$ for *j* ≠ 0, 3, 4, 5, 7, 8, 9.

The previous definition implies the following theorem.

Theorem 10.5.0. *If k* > 0, $\widetilde{IVLS}_{\tilde{S}}(j) = \widetilde{IVLS}_{\tilde{S}}(j^{-1})$.

Proof: The result follows since $k = \wedge\left\{\tilde{S}(s) \wedge \tilde{S}(t) \middle| \; s,t \in S \text{ and } int(s,t) = j\right\} =$

$\wedge\left\{\tilde{S}(s) \wedge \tilde{S}(t) \middle| \; s,t \in S \text{ and } int(t,s) = j^{-1}\right\}$. ♦

Section 10.6. Assigning membership values

Now that we have a language to describe hierarchical relations among musical elements, how do we assign the membership values? Theorists, such as Silberman, define a list of criteria that guides us in assigning membership values to pitch-classes in a piece of music.[19] For example, Silberman uses criteria such as rhythmic/metric placement of a pitch-class, register in which a pitch-class occurs, or how often a pitch-class occurs.[20] Other criteria can include such parameters as dynamics and timbre. For example, we can use the criterion of repetition and assign larger membership values to those pitch-classes that occur more often in a section of music. If we choose rhythmic placement as our criterion, we can assign higher membership values to those pitch-classes that occur on strong metrical beats. We can also combine and order the various criteria, so that, for example, a pitch-class satisfying both repetition and loud dynamics is valued more heavily than a pitch-class satisfying repetition only.

One important point should be made. Although some list of criteria will probably be necessary (from an analyst's or composer's point of view) to order the various elements in a piece of music, there is, from our perspective, no one fixed list. Each individual will select those criteria to organize a piece of music in ways that conform to that individual's understanding of the piece.

[19] Silberman 1997: 7-8

Section 10.7. The segmentation problem

This last point brings us to the segmentation problem. Pitch-class theory requires the selection of certain pitches in a piece of music to form pitch-class sets. But which particular pitches does one choose? More generally, the segmentation problem lies in the *procedure* for determining those musical elements of a piece of music to be analyzed.[21] Segmentation of a musical passage consists of carving it into interesting and relevant sets of musical elements. How can we begin?

In analyzing a piece of music, we want to consider many different segmentations of a musical passage before choosing the one that lends itself best to our understanding of that passage.[22] Many relevant segmentations of a single musical passage are possible, depending upon the criteria one chooses to relate a group of pitches together.[23] Forte and others have described lists of criteria to determine those relevant segmentations.[24]

Viewing pitch-class theory through the language of fuzzy set theory should cast the segmentation procedure in a more revealing light. Fuzzy pitch-class theory proceeds upon the very idea of segmentation, that is, subjectively determining the various membership values for each of the musical elements under analysis. Musical elements that are considered important and relevant for analytic purposes are those given nonzero membership values; i.e., those elements that are placed into sets.

Fuzzy pitch-class theory is direct about the segmentation procedure. Those elements that are not placed into sets, or not circled as sets in the musical score, are not considered less important because of some intrinsic property that they hold in relation to the piece of music. Rather, those elements have not been chosen for analysis because they have been subjectively assigned a membership value of 0 by the individual analyst.

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20 Ibid.
21 Forte 1973: 83; Hasty 1981: 54; Strauss 2000: 51; Benjamin 1974: 177
22 Strauss 2000: 52
23 Benjamin 1974: 178
24 Forte 1973: 83; Hasty 1981: 57-58; Strauss 2000: 51-52. For example, if a group of pitches are isolated through conventional means, such as separated by rests or beamed together rhythmically, then it makes good sense to place those pitches, as pitch-classes, into a pitch-class set. Forte calls these sets primary segments. Further, we can systematically consider all possible subsegments (subsets) of primary segments. Forte calls this process *imbrication*. Forte 1973: 83-84.

Viewed in this light, we can say that the segmentation problem is really a freedom for individual analysts to determine those aspects of a piece of music they deem important.